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Stability and Independence for Multivariate Refinable Distributions

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Due to their so-called time-frequency localization properties, wavelets have become a powerful tool in signal analysis and image processing. Typical constructions of wavelets depend on the stability of the shifts of an underlying refinable function. In this paper, we derive necessary and sufficient conditions for the stability of the shifts of certain compactly supported refinable functions. These conditions are in terms of the zeros of the refinement mask. Our results are actually applicable to more general distributions which are not of function type, if we generalize the notion of stability appropriately. We also provide a similar characterization of the (global) linear independence of the shifts. We present several examples illustrating our results, as well as one example in which known results on box splines are derived using the theorems of this paper. © 1999 Academic Press

1. INTRODUCTION

In this paper we present a characterization of the stability and linear independence of the shifts of certain compactly supported refinable functions in terms of the refinement mask. Our results are applicable to a large class of multivariate distributions which includes (but is not limited to) tensor products and box splines.

For $1 \leqslant p \leqslant \infty$, we denote by $L^p(\mathbb{R}^d)$ the set of all measurable functions $f\colon \mathbb{R}^d \to \mathbb{C}$ satisfying

$$\|f\|_{L^p} := \left\{ \int_{\mathbb{R}^d} |f(x)|^p \, dx \right\}^{1/p} < \infty.$$

We also denote by $\mathscr{D}'(\mathbb{R}^d)$ the set of all continuous linear functionals $\phi\colon \mathscr{D}(\mathbb{R}^d) \to \mathbb{C}$, where $\mathscr{D}(\mathbb{R}^d)$ is the set of all compactly supported infinitely differentiable functions with the standard topology (cf., e.g., [13, Chap. 6]).

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A function $\phi \in L^p(\mathbb{R}^d)$ is said to have *ℓ^p-stable shifts* if there exist positive constants C and D such that

$$C\,\|a\|_{\ell^p} \leqslant \Big\| \sum_{\alpha \in \mathbb{Z}^d} a(\alpha)\,\phi(\cdot - \alpha) \Big\|_{L^p} \leqslant D\,\|a\|_{\ell^p}$$

for all $a \in \ell^p(\mathbb{Z}^d)$ (it is often said that ϕ provides a *Riesz basis* in $L^p(\mathbb{R}^d)$ in this case); a compactly supported $\phi \in \mathscr{D}'(\mathbb{R}^d)$ is said to have *linearly independent shifts* if the map

$$\phi *' : \mathbb{C}^{\mathbb{Z}^d} \to \mathscr{D}'(\mathbb{R}^d) : a \mapsto \sum_{\alpha \in \mathbb{Z}^d} a(\alpha)\,\phi(\cdot - \alpha)$$

is one-to-one; and a function $\phi \in L^2(\mathbb{R}^d)$ is said to have *orthonormal shifts* if

$$\langle \phi, \phi(\cdot - \alpha) \rangle := \int_{\mathbb{R}^d} \phi(t)\,\overline{\phi(t-\alpha)}\,dt = \delta_\alpha := \begin{cases} 1, & \text{if } \alpha = 0; \\ 0, & \text{if } \alpha \in \mathbb{Z}^d \backslash 0. \end{cases}$$

It is worth pointing out at this time that, for the shifts of a compactly supported $\phi \in L^2(\mathbb{R}^d)$, orthonormality implies linear independence implies ℓ^p-stability for $1 \leqslant p \leqslant 2$.

All of these properties can be characterized in terms of the Fourier transform of ϕ. For example, if ϕ has orthonormal shifts, then

$$\begin{aligned} \delta_\alpha &= \int_{\mathbb{R}^d} \phi(t)\,\overline{\phi(t-\alpha)}\,dt = \int_{\mathbb{R}^d} |\hat{\phi}(\omega)|^2\, e^{i\langle \alpha, \omega \rangle}\, d\mu(\omega) \\ &= \int_{\mathbb{T}^d} \sum_{\beta \in \mathbb{Z}^d} |\hat{\phi}(\omega + 2\beta\pi)|^2\, e^{i\langle \alpha, \omega \rangle}\, d\mu(\omega), \end{aligned}$$

where $\mathbb{T}^d := [0, 2\pi)^d$ and $(2\pi)^d\, d\mu(\omega) := d\omega$. It follows that a compactly supported function $\phi \in L^2(\mathbb{R}^d)$ has orthonormal shifts if and only if

$$\sum_{\beta \in \mathbb{Z}^d} |\hat{\phi}(\cdot + 2\beta\pi)|^2 = 1.$$

As this paper deals with compactly supported distributions, $\hat{\phi}$ will be used to represent the Fourier–Laplace transform of ϕ, which is an entire function defined on all of $\mathbb{C}^d$ for all compactly supported $\phi \in \mathscr{D}'(\mathbb{R}^d)$.

In [7], Jia and Micchelli proved that a compactly supported $\phi \in L^p(\mathbb{R}^d)$ $(1 \leqslant p \leqslant \infty)$ has ℓ^p-stable shifts if and only if the set

$$N_{\mathbb{R}}(\phi) := \{\vartheta \in \mathbb{T}^d : \hat{\phi}(\vartheta + 2\alpha\pi) = 0\ \forall \alpha \in \mathbb{Z}^d\}$$

is empty. And it was proved by Ron in [12] that a compactly supported $\phi \in \mathscr{D}'(\mathbb{R}^d)$ has linear independent shifts if and only if the set

$$N_{\mathbb{C}}(\phi) := \{\vartheta \in \mathbb{T}^d + i\mathbb{R}^d \colon \hat{\phi}(\vartheta + 2\alpha\pi) = 0 \ \forall \alpha \in \mathbb{Z}^d\}$$

is empty.

Notice that the stability criterion, that $N_{\mathbb{R}}(\phi)$ be empty, is independent of p. For this reason, we will say that a compactly supported $\phi \in \mathscr{D}'(\mathbb{R}^d)$ *has suitable shifts* if $N_{\mathbb{R}}(\phi)$ is empty. If it does happen that ϕ is in $L^p(\mathbb{R}^d)$ for some p, then suitability and ℓ^p-stability are equivalent. In this paper, we investigate the suitability and linear independence of the shifts of compactly supported refinable distributions.

A compactly supported $\phi \in \mathscr{D}'(\mathbb{R}^d)$ is said to be *refinable* if (ϕ is not identically zero and) there exists a finitely supported sequence $a \colon \mathbb{Z}^d \to \mathbb{C}$ satisfying

$$\phi = \sum_{\alpha \in \mathbb{Z}^d} a(\alpha)\, \phi(2\cdot - \alpha).$$

Equivalently, ϕ is refinable if

$$\hat{\phi}(2\omega) = A(\omega)\, \hat{\phi}(\omega) \qquad \text{for all} \quad \omega \in \mathbb{C}^d, \tag{1.1}$$

where

$$A := \frac{1}{2^d} \sum_{\alpha \in \mathbb{Z}^d} a(\alpha)\, e^{-i\langle \cdot, \alpha \rangle}.$$

Equation (1.1) is called the *refinement equation* and we refer to the trigonometric polynomial A as the (*refinement*) *mask*. It is known (cf. [4]) that if $A(0) = 1$, then there exists a unique distributional solution to Eq. (1.1) with $\hat{\phi}(0) = 1$.

The characterizations given above for orthonormality, stability, and linear independence are all in terms of the Fourier–Laplace transform $\hat{\phi}$. However, for refinable ϕ it is actually more desirable to characterize these properties in terms of the mask A. Since, as is well known, a compactly supported refinable function $\phi \in L^2(\mathbb{R}^d)$ with mask A has orthonormal shifts if and only if ϕ has ℓ^2-stable shifts and

$$\sum_{\nu \in \{0,1\}^d} |A(\cdot + \nu\pi)|^2 = 1 \qquad \text{on } \mathbb{R}^d,$$

it is sufficient to characterize suitability and linear independence (assuming one has some way to ensure that $\phi \in L^p(\mathbb{R}^d)$ if necessary).

In the univariate case ($d=1$), suitability and linear independence of the shifts of a compactly supported distribution have been characterized in terms of the mask by Jia and Wang [8]. Their arguments relied on the fact (cf. [12]) that, for a non-zero compactly supported distribution $\phi \in \mathscr{D}'(\mathbb{R})$, the set $N_{\mathbb{C}}(\phi)$ is finite. Unfortunately, this statement is invalid for multivariate distributions.

To analyze the multivariate case, we consider distributions ϕ whose Fourier–Laplace transform $\hat{\phi}$ has the form

$$\hat{\phi} = \hat{\phi}_{\Xi} := \prod_{\xi \in \Xi} \hat{\phi}_{\xi}(\langle \cdot, \xi \rangle), \tag{1.2}$$

where Ξ is a finite subset of $\mathbb{Z}^d \backslash 0$ and, for each $\xi \in \Xi$, ϕ_{ξ} is a univariate distribution of compact support. This defines a compactly supported distribution $\phi \in \mathscr{D}'(\mathbb{R}^d)$.

It is important to note that, with this definition, each ϕ_{ξ} is univariate, while $\hat{\phi}_{\{\xi\}} = \hat{\phi}_{\xi}(\langle \cdot, \xi \rangle)$ defines $\phi_{\{\xi\}}$ as an element of $\mathscr{D}'(\mathbb{R}^d)$ (with support in the line $\mathbb{R}\xi$). We also point out that if ϕ_{ξ} is refinable for every $\xi \in \Xi$, say with mask A_{ξ}, then ϕ_{Ξ} is also refinable with mask

$$A_{\Xi} := \prod_{\xi \in \Xi} A_{\xi}(\langle \cdot, \xi \rangle).$$

It is clear from (1.2) and the characterization of suitability (resp. linear independence) in terms of the set $N_{\mathbb{R}}(\phi)$ (resp. $N_{\mathbb{C}}(\phi)$) that, if the shifts of ϕ_{Ξ} are suitable (resp. linearly independent), then the shifts of ϕ_Y must be suitable (resp. linearly independent) for every $Y \subset \Xi$.

Now, suppose $\phi = \phi_{\Xi}$ is of the type (1.2) and suppose $Y \subset \Xi$ satisfies $d_Y := \dim \operatorname{span} Y < d$. Then $\hat{\phi}_Y$ is constant in directions orthogonal to Y. Therefore, if $N_{\mathbb{R}}(\phi_Y)$ is non-empty, say $\vartheta \in N_{\mathbb{R}}(\phi_Y)$, then for any $\eta \in Y^{\perp}$, $\hat{\phi}_Y(\vartheta + \eta + 2\alpha\pi)$ is zero for all $\alpha \in \mathbb{Z}^d$; i.e., the set $N_{\mathbb{R}}(\phi_Y)$ is infinite. The main results of this paper are based on the converse of this, namely:

Lemma 1.1. *If $\phi = \phi_{\Xi} \in \mathscr{D}'(\mathbb{R}^d)$ is of the type* (1.2), *and if $N_{\mathbb{R}}(\phi)$ is infinite, then there is some $Y \subset \Xi$ with $d_Y = \dim \operatorname{span} Y < d$ so that $N_{\mathbb{R}}(\phi_Y)$ is already non-empty.*

It should be noted that this lemma does not require that ϕ be refinable.

This lemma will actually lead to a complete characterization of suitability and linear independence in terms of the mask for refinable distributions of the type (1.2). If the shifts of ϕ_{Ξ} are not suitable and $N_{\mathbb{R}}(\phi_{\Xi})$ is infinite, for example, then any minimal $Y \subset \Xi$ with $N_{\mathbb{R}}(\phi_Y) \neq \{\ \}$ will satisfy $d_Y < d$ by Lemma 1.1. In this case, ϕ_Y actually has its support in the subspace spanned by Y, and the map $\phi *'$ is not even bounded below when restricted to $\ell^2(\mathbb{Z}^d \cap \operatorname{span} Y)$. We may then analyze those shifts of ϕ_Y with support in span Y. Equivalently, we may analyze the set $N_{\mathbb{R}}(\phi_Y) \cap \operatorname{span} Y$ which, since Y is minimal, must be finite. This reasoning, which will be more rigorously presented later, works for suitability. To handle linear independence, we use the following result due to Zhou:

Result 1.2 [15]. Suppose the compactly supported distribution ϕ is refinable with mask A. Then ϕ has linearly independent shifts if and only if ϕ has suitable shifts and A has no π-periodic zeros in $\mathbb{C}^d$.

2. STATEMENT OF MAIN RESULTS

From this point forward we assume only that ϕ is a compactly supported distribution. If we refer to $\phi = \phi_\Xi$, then we are also assuming that ϕ is of the type (1.2). We make no assumption that $\phi \in L^p(\mathbb{R}^d)$, or even of function type at all. Nor do we assume that ϕ is necessarily refinable (if this is needed, it will be stated explicitly). However, when we do refer to a refinable ϕ, we will assume that $A(0) = 1$ and that ϕ is the distributional solution to Eq. (1.1) with $\hat{\phi}(0) = 1$.

In the statement of results that follows, and throughout this paper, we will say that the function A has a

- *π-periodic zero in* $\mathbb{R}^d$ (*resp.* $\mathbb{C}^d$) if there exists $z \in \mathbb{R}^d$ (resp. $\mathbb{C}^d$) such that

$$A(z + \alpha\pi) = 0 \qquad \text{for all} \quad \alpha \in \mathbb{Z}^d;$$

- *contaminating zero in* $\mathbb{R}^d$ if there exists an integer $m \geqslant 2$ and $\mu \in \mathbb{Z}^d \backslash (2^m - 1)\, \mathbb{Z}^d$ such that

$$A\left(2^k \frac{2\mu\pi}{2^m - 1} + \nu\pi\right) = 0 \qquad \text{for all} \quad \nu \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d, \qquad k \in \{0, 1, 2, ...\}.$$

Equivalent definitions follow from the 2π-periodicity of A. Namely, A has a

- π-periodic zero in $\mathbb{R}^d$ if there exists $x \in \mathbb{T}^d$ such that $A(x + \alpha\pi) = 0$ for all $\alpha \in \{0, 1\}^d$;
- π-periodic zero in $\mathbb{C}^d$ if there exists $z \in \mathbb{T}^d + i\mathbb{R}^d$ such that $A(z + \alpha\pi) = 0$ for all $\alpha \in \{0, 1\}^d$;
- contaminating zero in $\mathbb{R}^d$ if there is an integer $m \geqslant 2$ and $\mu \in \{0, 1, ..., 2^m - 2\}^d \backslash 0$ such that $A(2k(2\mu\pi)/(2^m - 1) + \nu\pi) = 0$ for all $\nu \in \{0, 1\}^d \backslash 0$, $k \in \{0, 1, 2, ..., m - 1\}$.

The following two theorems will be proved using the arguments of [8]. Note that we assume the relevant set, $N_{\mathbb{R}}(\phi)$ or $N_{\mathbb{C}}(\phi)$, to be finite, whereas this assumption is not explicit in the statement of results in [8]. It is, however, implied by the fact that, for univariate ϕ, $N_{\mathbb{C}}(\phi)$ is always finite.

THEOREM 2.1. *Suppose ϕ is refinable and $N_{\mathbb{R}}(\phi)$ is finite. Then the shifts of ϕ are suitable if and only if the refinement mask A satisfies*

(i) *A has no π-periodic zeros in $\mathbb{R}^d$, and*

(ii) *A has no contaminating zeros in $\mathbb{R}^d$.*

THEOREM 2.2. *Suppose ϕ is refinable and $N_{\mathbb{C}}(\phi)$ is finite. Then the shifts of ϕ are linearly independent if and only if the refinement mask A satisfies*

(i) *A has no π-periodic zeros in $\mathbb{C}^d$; and*

(ii) *A has no contaminating zeros in $\mathbb{R}^d$.*

The assumption that $N_{\mathbb{R}}(\phi)$ or $N_{\mathbb{C}}(\phi)$ be finite will only be used to prove the sufficiency. We therefore have the following

THEOREM 2.3. *Suppose ϕ is refinable. If the shifts of ϕ are suitable (resp. linearly independent), then*

(i) *A has no π-periodic zeros in $\mathbb{R}^d$ (resp. $\mathbb{C}^d$) and*

(ii) *A has no contaminating zeros in $\mathbb{R}^d$.*

It should be noted that none of the theorems up to this point in this section required that ϕ be of the type (1.2).

Unfortunately, the assumption that $N_{\mathbb{R}}(\phi)$ or $N_{\mathbb{C}}(\phi)$ be finite in Theorems 2.1 and 2.2 cannot be easily verified in terms of the mask. Moreover, Example 4.1 shows that this assumption cannot be eliminated in general. We will, however, eliminate it for distributions of the type (1.2), under the mild conditions that dim span $\Xi = d$ and $A_\xi(\pi) = 0$ for all $\xi \in \Xi$.

THEOREM 2.4. *Suppose $\phi = \phi_\Xi$ of type (1.2) is refinable. Suppose* dim span $\Xi = d$ *and $A_\xi(\pi) = 0$ for every $\xi \in \Xi$. Then the shifts of ϕ are suitable (resp. linearly independent) if and only if the refinement mask $A := A_\Xi$ satisfies*

(i) *A has no π-periodic zeros in $\mathbb{R}^d$ (resp. $\mathbb{C}^d$), and*

(ii) *A has no contaminating zeros in $\mathbb{R}^d$.*

By assuming that Ξ spans, we are merely assuming that the support of ϕ_Ξ is not contained in some lower dimensional subspace. We should also point out that the easily verifiable assumption that $A_\xi(\pi)$ be zero is very reasonable. For example, the proof of Theorem 2.4 from [7] shows that $\hat{\phi}_\xi$ vanishes on the set $2\mathbb{Z}\pi\backslash 0$ whenever $\hat{\phi}_\xi$ vanishes at infinity. If the shifts of ϕ_ξ are suitable, this in turn implies that $A_\xi(\pi)$ is zero. The condition that $\hat{\phi}$ vanishes at infinity is satisfied, for example, for compactly supported $\phi \in L^p(\mathbb{R})$ with $1 \leqslant p \leqslant \infty$. These observations lead immediately to the

COROLLARY 2.5. *Suppose* dim span $\Xi = d$. *For each* $\xi \in \Xi$, *let* $\phi = \phi_\xi$ *be the solution to Eq.* (1.1) *with mask* $A = A_\xi$. *If* $\phi_\xi \in L^p(\mathbb{R})$ *for each* $\xi \in \Xi$, *then the shifts of* ϕ_Ξ *are suitable* (*resp. linearly independent*) *if and only if*

(i) A_Ξ *has no* π-*periodic zeros in* $\mathbb{R}^d$ (*resp.* $\mathbb{C}^d$),

(ii) A_Ξ *has no contaminating zeros in* $\mathbb{R}^d$, *and*

(iii) $A_\xi(\pi) = 0$ *for every* $\xi \in \Xi$.

Remark. It can be shown that if $\phi_\xi \in L^p(\mathbb{R})$ and $A_\xi(\pi) \neq 0$ then A_Ξ has a π-periodic zero in $\mathbb{R}^d$. In other words, the three conditions in Corollary 2.5 are actually equivalent to just the first two.

As already pointed out, the necessity of the conditions in Theorem 2.4 still holds for compactly supported refinable functions not of type (1.2). A natural question that arises is whether these conditions are still sufficient (say under the assumption that $\phi \in L^1(\mathbb{R}^d)$). It is clear from Theorem 2.2 and Result 1.2, that if a refinable function ϕ with mask A has dependent shifts while A has no π-periodic or contaminating zeros, then $N_{\mathbb{R}}(\phi)$ must be infinite. This would be the case, for instance, if ϕ were of the form $\hat{\phi} = \hat{\phi}_1 \hat{\phi}_2$ where ϕ_1, say, did not have suitable shifts and was supported in some lower dimensional subspace. In fact, we are not aware of any refinable function ϕ for which $N_{\mathbb{R}}(\phi)$ is infinite and which is not of this form (though we make no conjecture that none exists). Under some assumptions on ϕ_2, it is likely that the arguments of this paper could be generalized to handle this (slightly) more general situation.

The proof of Theorem 2.4 will be facilitated by the following

LEMMA 2.6. *Suppose* dim span $\Xi = d$ *and that* $A_\xi(\pi) = 0$ *for every* $\xi \in \Xi$. *If there is a basis* $B \subset \Xi$ *with* det $B \in 2\mathbb{Z}$ *then* A_Ξ *has a* π-*periodic zero in* $\mathbb{R}^d$. *If some basis* $B \subset \Xi$ *satisfies* det $B \neq \pm 1$ *and* det $B \notin 2\mathbb{Z}$ *then* A_Ξ *has a contaminating zero in* $\mathbb{R}^d$.

Already, Lemma 2.6 together with Theorem 2.3 provides a proof of the known

Result 2.7. Suppose dim span $\Xi = d$ and that $A_\xi(\pi) = 0$ for every $\xi \in \Xi$. If the shifts of ϕ_Ξ are suitable, then $|\det B| = 1$ for every basis $B \subset \Xi$.

However, the condition $|\det B| = 1$ for all bases is not sufficient. Example 4.2 illustrates a situation in which $\dim \operatorname{span} \Xi = d$, each ϕ_ξ has linearly independent shifts, each A_ξ has a zero at π, and $|\det B| = 1$ for all bases $B \subset \Xi$; yet the shifts of ϕ_Ξ are not even suitable.

We would like to point out that, in [9], Lawton *et al.* provided a characterization of orthonormality based on the 1-eigenspace of the operator

$$L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)\colon f \mapsto \sum_{\nu \in \{0,1\}^d} |A(\cdot + \nu\pi)|^2 \, f(\cdot + \nu\pi).$$

In [10, Sect. 4], Long and Chen provided similar criteria for the related property of biorthogonality. Their paper also provided a multivariate version of the so-called Cohen conditions. These results require that $\phi \in L^2(\mathbb{R}^d)$. Moreover, the conditions provided can be very difficult to check. In this paper, we present conditions which we believe to be simpler and which are valid even for $\phi \notin L^2(\mathbb{R}^d)$.

3. PROOF OF MAIN RESULTS

A fair portion of our analysis will involve the Smith normal form of an integral matrix (cf. e.g., [11, pp. 26–28]):

Smith Normal Form. Every matrix $Y \in \mathbb{Z}^{d \times n}$ has the form

$$Y = UDV$$

with $U \in \mathbb{Z}^{d \times d}$ and $V \in \mathbb{Z}^{n \times n}$ satisfying $\det U = \det V = 1$, and $D \in \mathbb{Z}^{d \times n}$ satisfying $D_{ij} \neq 0$ if and only if $i = j \leqslant \operatorname{rank} Y$.

More specifically, we will use the

COROLLARY 3.1. *For any finite $Y \subset \mathbb{Z}^d$ with $d_Y = \dim \operatorname{span} Y$, there exists $U \in \mathbb{Z}^{d \times d_Y}$ and $V = \{v(y)\colon y \in Y\} \subset \mathbb{Z}^{d_Y}$ such that*

(i) $Y = UV$ (*i.e.* $y = Uv(y)$ *for each* $y \in Y$),

(ii) $\mathbb{Z}^d U = \mathbb{Z}^{d_Y}$.

Proof. For $Y \in \mathbb{Z}^{d \times n}$ with Smith normal form $\tilde{U}\tilde{D}\tilde{V}$, define $U \in \mathbb{Z}^{d \times d_Y}$ and $V \in \mathbb{Z}^{d_Y \times n}$ by

$$U_{ij} := \tilde{U}_{ij}, \qquad 1 \leqslant i \leqslant d, \qquad 1 \leqslant j \leqslant d_Y,$$

and

$$V_{ij} := \tilde{D}_{ii} \tilde{V}_{ij}, \qquad 1 \leqslant i \leqslant d_Y, \qquad 1 \leqslant j \leqslant n.$$

Then, $Y = UV$ and $\mathbb{Z}^d U = \mathbb{Z}^{d_Y}$. The sets Y and V of Corollary 3.1 are the column sets of the matrices Y and V here. ▮

The arguments in the proof of Theorems 2.1 and 2.2 are based on arguments given in [8]. The proof will depend on the following lemma, the univariate version of which also appears in [8].

LEMMA 3.2. *Suppose the compactly supported distribution ϕ is refinable with mask A. Suppose further that $N_{\mathbb{R}}(\phi)$ is finite. If A has no π-periodic zeros in $\mathbb{R}^d$, then every element of $N_{\mathbb{R}}(\phi)$ is of the form*

$$z = \frac{2\mu\pi}{2^m - 1}$$

for some integer $m \geqslant 2$ and some $\mu \in \{0, 1, ..., 2^m - 2\} \backslash 0$.

Proof. Let z be an element of $N_{\mathbb{R}}(\phi)$. Then $\hat{\phi}(z + 2\beta\pi) = 0$ for every $\beta \in \mathbb{Z}^d$. In particular, z is not in $2\pi\mathbb{Z}^d$, since $\hat{\phi}(0) = 1$. By Eq. (1.1),

$$0 = \hat{\phi}(z + 2\alpha\pi + 4\beta\pi) = A(z/2 + \alpha\pi)\, \hat{\phi}(z/2 + \alpha\pi + 2\beta\pi)$$

for all $\alpha, \beta \in \mathbb{Z}^d$. Our hypotheses ensure that $A(z/2 + \alpha\pi) \neq 0$ for some $\alpha \in \{0, 1\}^d$. Hence, for this α, we have $z/2 + \alpha\pi \in N_{\mathbb{R}}(\phi)$.

Let $z_0 := z$, $z_1 := z/2 + \alpha\pi$. Then $z_0, z_1 \in N_{\mathbb{R}}(\phi)$ and $2z_1 - z_0 \in 2\pi\mathbb{Z}^d$. Since z_1 is again in $N_{\mathbb{R}}(\phi)$, we can repeat the process ad infinitum to obtain a sequence $z_0, z_1, ...$ satisfying $z_k \in N_{\mathbb{R}}(\phi)$ and $2^k z_k - z_0 \in 2\pi\mathbb{Z}^d$ all k. Since $N_{\mathbb{R}}(\phi)$ is finite, we must have $z_k = z_l$ for some $l > k$. Then z_0 is equal to $2^k z_k + 2\mu_k\pi$ for some $\mu_k \in \mathbb{Z}^d$, and $z_0 = 2^l z_l + 2\mu_l\pi = 2^m 2^k z_k + 2\mu_l\pi$ for some $\mu_l \in \mathbb{Z}^d$ where $m := l - k$. Combining these two we get $z = z_0 = 2(2^m\mu_k - \mu_l)\,\pi/(2^m - 1) = 2\mu\pi/(2^m - 1)$ where $\mu := 2^m\mu_k - \mu_l \in \mathbb{Z}^d \backslash (2^m - 1)\,\mathbb{Z}^d$ since $z \notin 2\pi\mathbb{Z}^d$. In particular, $m \neq 1$. ▮

Proof of Theorems 2.1 *and* 2.2. It follows immediately from Eq. (1.1) that, if A has a π-periodic zero in $\mathbb{C}^d$, say $A(z + \alpha\pi) = 0$ for all $\alpha \in \mathbb{Z}^d$, then $2z \in N_{\mathbb{C}}(\phi)$ and the shifts of ϕ are not linearly independent. If $z \in \mathbb{R}^d$, then $2z \in N_{\mathbb{R}}(\phi)$ and the shifts of ϕ are not suitable.

Next we show that the shifts of ϕ are not suitable (hence not linearly independent) if A has a contaminating zero in $\mathbb{R}^d$. Suppose the integer $m \geqslant 2$ and $\mu \in \{0, 1, ..., 2^m - 2\}^d \backslash 0$ satisfy

$$A\left(2^k \frac{2\mu\pi}{2^m - 1} + \nu\pi\right) = 0 \qquad \text{for all} \quad k \in \{0, 1, 2, ...\},\ \nu \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d. \tag{3.1}$$

We claim that $2\mu\pi/(2^m - 1) \in N_{\mathbb{R}}(\phi)$.

We observe from Eq. (1.1) that

$$\hat{\phi} = \hat{\phi}(\cdot/2)\, A(\cdot/2) = \hat{\phi}(2^{-n}\cdot) \prod_{j=1}^{n} A(2^{-j}\cdot).$$

So $A(z)=0$ implies $\hat{\phi}(2^j z)=0$ for every $j \in \{1, 2, 3, ...\}$. Since we are assuming (3.1), we may show that $\hat{\phi}(2\mu\pi/(2^m-1)+2\alpha\pi)=0$ by finding $j \in \{1, 2, 3, ...\}$, $k \in \{0, 1, 2, ...\}$, and $v \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d$ so that

$$\frac{2\mu\pi}{2^m-1} + 2\alpha\pi = 2^j \left(2^k \frac{2\mu\pi}{2^m-1} + v\pi \right)$$

or equivalently,

$$\mu + (2^m-1)\,\alpha = 2^{j-1}(2^{k+1}\mu + (2^m-1)\, v).$$

Since $\mu \notin (2^m-1)\,\mathbb{Z}^d$ (hence $\mu + (2^m-1)\,\alpha \neq 0$), we can write $\mu + (2^m-1)\,\alpha = 2^{j-1}\beta$ with $j \in \{1, 2, 3, ...\}$ and $\beta \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d$. Now choose $n \in \mathbb{Z}$ so that $j \leqslant mn < j+m$ and define $k := mn - j$, then $k \in \{0, 1, 2, ...\}$ and

$$\begin{aligned}
\beta &= 2^{mn-j+1} 2^{j-1}\beta - (2^{mn}-1)\,\beta \\
&= 2^{mn-j+1}(\mu + (2^m-1)\,\alpha) - (2^m-1)(2^{m(n-1)} + \cdots + 2^m + 1)\,\beta \\
&= 2^{k+1}\mu + (2^m-1)(2^{k+1}\alpha - (2^{m(n-1)} + \cdots + 2^m + 1)\,\beta) \\
&= 2^{k+1}\mu + (2^m-1)\, v,
\end{aligned}$$

where $v := 2^{k+1}\alpha - (2^{m(n-1)} + \cdots 2^m + 1)\,\beta \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d$. This completes the proof that the shifts of ϕ are not suitable if A has a contaminating zero.

By Result 1.2, it is sufficient to complete the proof only for Theorem 2.1. We assume that A has no π-periodic zero in $\mathbb{R}^d$ and that $N_{\mathbb{R}}(\phi)$ is (finite but) not empty; and we show that A must then have a contaminating zero in $\mathbb{R}^d$. This will be simpler if we work not with $N_{\mathbb{R}}(\phi)$, but introduce instead the set

$$N^+ := N_{\mathbb{R}}(\phi) + 2\pi\mathbb{Z}^d = \{\vartheta \in \mathbb{R}^d \colon \hat{\phi}(\vartheta + 2\alpha\pi) = 0\ \forall \alpha \in \mathbb{Z}^d\}.$$

We begin by showing that, for any integer $m \geqslant 2$ and $\mu \in \mathbb{Z}^d \backslash (2^m-1)\,\mathbb{Z}^d$,

$$\frac{4\mu\pi}{2^m-1} \in N^+ \Rightarrow \begin{cases} \dfrac{2\mu\pi}{2^m-1} \in N^+ & \text{and} \\ A\left(\dfrac{2\mu\pi}{2^m-1} + v\pi\right) = 0 & \text{for all} \quad v \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d. \end{cases} \tag{3.2}$$

Notice that, given $\nu \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d$, $2\mu\pi/(2^m - 1) + \nu\pi$ is not of the form $2\mu'\pi/(2^{m'} - 1)$ for any integer $m' \geqslant 2$ and $\mu' \in \mathbb{Z}^d \backslash (2^{m'} - 1)\,\mathbb{Z}^d$, hence by Lemma 3.2,

$$\frac{2\mu\pi}{2^m - 1} + \nu\pi \notin N^+ \qquad \text{for any} \quad \nu \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d. \tag{3.3}$$

Now fix $\nu \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d$. Since $4\mu\pi/(2^m - 1) \in N^+$,

$$0 = \hat{\phi}\left(\frac{4\mu\pi}{2^m - 1} + 2\nu\pi + 4\alpha\pi\right) = \hat{\phi}\left(\frac{2\mu\pi}{2^m - 1} + \nu\pi + 2\alpha\pi\right) A\left(\frac{2\mu\pi}{2^m - 1} + \nu\pi\right)$$

for all $\alpha \in \mathbb{Z}^d$. This, together with (3.3), yields

$$A\left(\frac{2\mu\pi}{2^m - 1} + \nu\pi\right) = 0.$$

Since $\nu \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d$ was arbitrary and we have assumed that A has no π-periodic zeros, it follows that

$$A\left(\frac{2\mu\pi}{2^m - 1}\right) \neq 0. \tag{3.4}$$

Again, $4\mu\pi/(2^m - 1) \in N^+$ implies that for all $\alpha \in \mathbb{Z}^d$,

$$0 = \hat{\phi}\left(\frac{4\mu\pi}{2^m - 1} + 4\alpha\pi\right) = \hat{\phi}\left(\frac{2\mu\pi}{2^m - 1} + 2\alpha\pi\right) A\left(\frac{2\mu\pi}{2^m - 1}\right)$$

which, together with (3.4), yields

$$\frac{2\mu\pi}{2^m - 1} \in N^+,$$

proving the claim (3.2).

Now, by Lemma 3.2, if A has no π-periodic zeros in $\mathbb{R}^d$ and $N_{\mathbb{R}}(\phi)$ is finite and nonempty, then $N_{\mathbb{R}}(\phi)$ contains a point of the form $2\mu\pi/(2^m - 1)$ where $m \geqslant 2$ and $\mu \in \mathbb{Z}^d \backslash (2^m - 1)\,\mathbb{Z}^d$. In fact, since $2^{m+1}\mu\pi/(2^m - 1) - 2\mu\pi/(2^m - 1) = 2\mu\pi$, we actually have

$$4\,\frac{2^{m-1}\mu\pi}{2^m - 1} \in N^+.$$

Applying (3.2) repeatedly, we see that for $k \in \{0, 1, 2, ..., m-1\}$,

$$A\left(2^k \frac{2\mu\pi}{2^m - 1} + \nu\pi\right) = 0 \qquad \text{for all} \quad \nu \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d;$$

i.e., A has a contaminating zero in $\mathbb{R}^d$. ∎

The proof of Lemma 1.1 will require the following definitions. For $Y \subset \Xi$, we define $\langle Y \rangle := (\text{span } Y) \cap \Xi$; for $X \subset \Xi$ with linearly independent elements and $a \in \mathbb{R}^X$, we define

$$S_{X,a} := \{z \in \mathbb{R}^d : \langle z, \xi \rangle = a(\xi) \ \forall \xi \in X\};$$

and for $\xi \in \Xi$, we define

$$K_\xi := \{\langle z, \xi \rangle : z \in N_{\mathbb{R}}(\phi_\Xi)\}.$$

So, $\langle Y \rangle$ is a subset of Ξ, $S_{X,a}$ is an affine subspace of $\mathbb{R}^d$ of dimension $d - d_X = d - \# X$ (recall $d_X := \dim \text{span } X$), and K_ξ is a subset of $\mathbb{R}$. Note that if $X \subset \Xi$ is empty, then $\mathbb{R}^X$ is a zero-dimensional vector space consisting of only one element which we call 0. In this case, the set $S_{\{\},0}$ is the entire space $\mathbb{R}^d$.

Proof of Lemma 1.1. The statement is trivial if $\dim \text{span } \Xi < d$, so we assume throughout that $\dim \text{span } \Xi = d$. We also assume that $N_{\mathbb{R}}(\phi_Y)$ is empty for any $Y \subset \Xi$ satisfying $d_Y < d$. We then show that $N_{\mathbb{R}} := N_{\mathbb{R}}(\phi)$ is finite. This is the case $X = \{\ \}$, $a = 0$ of the

Claim 3.3. For any $X \subset \Xi$ with linearly independent elements and any $a \in \mathbb{R}^X$, the set $N_{\mathbb{R}} \cap S_{X,a}$ is finite.

The proof is by induction on $d - d_X$ (with a arbitrary). To begin, we show that the claim is valid when $d_X = d$. In this case, $S_{X,a}$ consists of a single point, so of course $N_{\mathbb{R}} \cap S_{X,a}$ is finite.

Now suppose that $X \subset \Xi$ consists of m linearly independent vectors with $d_X = m < d$ and that $a \in \mathbb{R}^X$ is given. Further, assume that for any $\tilde{X} \subset \Xi$ consisting of $d_{\tilde{X}} = m + 1$ linearly independent vectors and any $\tilde{a} \in \mathbb{R}^{\tilde{X}}$, the set $N_{\mathbb{R}} \cap S_{\tilde{X},\tilde{a}}$ is finite. We want to show that $N_{\mathbb{R}} \cap S_{X,a}$ is also finite.

To this end, let $z \in N_{\mathbb{R}} \cap S_{X,a}$ be given. Since $d_{\langle X \rangle} = d_X < d$, we know that $N_{\mathbb{R}}(\phi_{\langle X \rangle})$ is empty and there exists $\alpha \in \mathbb{Z}^d$ so that $\hat{\phi}_{\langle X \rangle}(z + 2\alpha\pi) \neq 0$. In fact, since $\hat{\phi}_{\langle X \rangle}$ is constant on sets of the form $S_{X,a}$, $\hat{\phi}_{\langle X \rangle}(z + 2\alpha\pi)$ is non-zero for this α and any z in $N_{\mathbb{R}} \cap S_{X,a}$.

Now, since z is in $N_{\mathbb{R}}$, we must have $\hat{\phi}_{\Xi}(z+2\alpha\pi)=0$. So $\hat{\phi}_{\xi}(\langle z+2\alpha\pi, \xi\rangle)=0$, for some $\xi\in\Xi\backslash\langle X\rangle$. That is, any z in $N_{\mathbb{R}}\cap S_{X,a}$ is actually an element of

$$\bigcup_{\xi\in\Xi\backslash\langle X\rangle}\{\vartheta\in N_{\mathbb{R}}\cap S_{X,a}:\langle\vartheta,\xi\rangle\in\mathscr{Z}(\hat{\phi}_{\xi})-2\pi\langle\alpha,\xi\rangle\},$$

where $\mathscr{Z}(\hat{\phi}_{\xi}):=\{z\in\mathbb{C}:\hat{\phi}_{\xi}(z)=0\}$. Equivalently,

$$N_{\mathbb{R}}\cap S_{X,a}\subset\bigcup_{\xi\in\Xi\backslash\langle X\rangle}\ \bigcup_{u\in U(\xi)}(N_{\mathbb{R}}\cap S_{\tilde{X}_{\xi},\tilde{a}_{\xi,u}}),\tag{3.5}$$

where

$$\begin{aligned}U(\xi)&:=K_{\xi}\cap(\mathscr{Z}(\hat{\phi}_{\xi})-2\pi\langle\alpha,\xi\rangle),\\ \tilde{X}_{\xi}&:=X\cup\{\xi\},\\ \tilde{a}_{\xi,u}(\eta)&:=a(\eta)\qquad\text{for}\quad\eta\in X,\end{aligned}$$

and

$$\tilde{a}_{\xi,u}(\xi):=u.$$

Since the set $N_{\mathbb{R}}\subset\mathbb{T}^{d}$ is bounded, each K_{ξ} is also bounded. As each $\hat{\phi}_{\xi}$ is entire, the sets $\mathscr{Z}(\hat{\phi}_{\xi})$ are locally finite (i.e., $K\cap\mathscr{Z}(\hat{\phi}_{\xi})$ is finite for any bounded $K\subset\mathbb{C}$). We see therefore that the sets $U(\xi)$ are each finite.

In (3.5), each set in the unison is finite by the induction hypothesis and of course $\Xi\backslash\langle X\rangle$ is finite. The union is thus finite, and the claim and theorem are proved. ∎

In order to prove Lemma 2.6, we will need the following lemmata:

LEMMA 3.4. *For $B\in\mathbb{Z}^{d\times d}$ and $n\in\mathbb{Z}$, if the determinant of B divides n then the set $n\mathbb{Z}^{d}$ is a subset of $\mathbb{Z}^{d}B=\{\alpha B:\alpha\in\mathbb{Z}^{d}\}$.*

Proof. We use the Smith normal form, $B=UDV$, where U and V map $\mathbb{Z}^{d}$ one-to-one onto $\mathbb{Z}^{d}$, and D is a diagonal matrix. Since $\det B=\prod_{i=1}^{d}D_{ii}$ and $\det B$ divides n, D_{ii} divides n for all i. This implies that $n\mathbb{Z}^{d}\subset\mathbb{Z}^{d}D=\{\alpha D:\alpha\in\mathbb{Z}^{d}\}$, which in turn implies that $n\mathbb{Z}^{d}\subset\mathbb{Z}^{d}B$ since $n\mathbb{Z}^{d}=n\mathbb{Z}^{d}V$ and $\mathbb{Z}^{d}B=\mathbb{Z}^{d}UDV=\mathbb{Z}^{d}DV$. ∎

LEMMA 3.5. *For $B\in\mathbb{Z}^{d\times d}$, if $vB\in 2\mathbb{Z}^{d}$ for any $v\in\mathbb{Z}^{d}\backslash 2\mathbb{Z}^{d}$ then the determinant of B is even.*

Proof. We use the Smith normal form, $B=UDV$, where U and V map $\mathbb{Z}^{d}$ one-to-one onto $\mathbb{Z}^{d}$, and $D\in\mathbb{Z}^{d\times d}$ is a diagonal matrix.

Suppose we have $v \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d$ and $vB = vUDV \in 2\mathbb{Z}^d$. Then we have $vU \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d$ while $vUD \in 2\mathbb{Z}^d$. This implies that $D_{ii} \in 2\mathbb{Z}$ for some i, which proves the claim since $\det B = \prod_{i=1}^d D_{ii}$. ∎

Proof of Lemma 2.6. Suppose the basis $B \subset \Xi$ satisfies $\det B \in 2\mathbb{Z}$. We work with the field $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$, and we consider the matrix, $\tilde{B} \in \mathbb{Z}_2^{d \times d}$, obtained from B by the canonical projection of its entries onto $\mathbb{Z}_2$. Then $\det B \in 2\mathbb{Z} \Rightarrow \det \tilde{B} = 0$. So the map $\mathbb{Z}_2^d \to \mathbb{Z}_2^d : x \mapsto x\tilde{B}$ is not onto. I.e., there exists $w \in \mathbb{Z}_2^d$ so that every $\alpha \in \mathbb{Z}^d$ satisfies $w \neq \tilde{\alpha}\tilde{B} = \widetilde{\alpha B}$. Since B is a basis, we can find $z \in \mathbb{R}^d$ so that $\widetilde{zB} = w\pi$. We see that for any $\alpha \in \mathbb{Z}^d$, $((z/\pi + \alpha) B)^{\sim} = (w + \widetilde{\alpha B}) \neq 0$. Equivalently, for any $\alpha \in \mathbb{Z}^d$, some $\xi \in B \subset \Xi$ satisfies $\langle z + \alpha\pi, \xi \rangle \in (\mathbb{Z} \backslash 2\mathbb{Z})\, \pi$. Since A_ξ is 2π-periodic and $A_\xi(\pi) = 0$, this implies that $A_{\{\xi\}}(z + \alpha\pi) = A_\xi(\langle z + \alpha\pi, \xi \rangle) = 0$. Since $\alpha \in \mathbb{Z}^d$ was arbitrary, we see that $A_\Xi = \prod_{\xi \in \Xi} A_{\{\xi\}}$ has a π-periodic zero in $\mathbb{R}^d$.

Now suppose that $|\det B| \neq 1$ and $\det B \notin 2\mathbb{Z}$. Fermat's Little Theorem guarantees an integer $m \geqslant 2$ with

$$2^m \equiv 1 \qquad \mathrm{mod} \det B.$$

Since $|\det B| \neq 1$, the set $\mathbb{Z}^d B := \{\alpha B : \alpha \in \mathbb{Z}^d\}$ is not all of $\mathbb{Z}^d$. Let $\alpha \in \mathbb{Z}^d \backslash (\mathbb{Z}^d B)$. Then, since $\det B$ divides $2^m - 1$, hence $(2^m - 1)\, \mathbb{Z}^d \subset \mathbb{Z}^d B$ by Lemma 3.4, there is some $\mu \in \mathbb{Z}^d$ so that $(2^m - 1)\, \alpha = \mu B$. And since $\alpha \notin \mathbb{Z}^d B$, we must have $\mu \notin (2^m - 1)\, \mathbb{Z}^d$. So for this $\mu \in \mathbb{Z}^d \backslash (2^m - 1)\, \mathbb{Z}^d$ and any $k \in \{0, 1, 2, ..., m-1\}$ we have $2^k (2\mu)/(2^m - 1)\, B = 2^{k+1} \alpha \in 2\mathbb{Z}^d$. Finally, the fact that $\det B$ is odd, along with Lemma 3.5 implies that

$$\left(2^k \frac{2\mu}{2^m - 1} + v\right) B \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d \qquad \text{for all} \quad k \in \{0, 1, 2, ..., m-1\}, \quad v \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d.$$

Since $A_\xi((2\mathbb{Z} + 1)\, \pi) = \{0\}$ for each $\xi \in B$, and $A_B(z) = 0 \Rightarrow A_\Xi(z) = 0$, we see that A_Ξ has a contaminating zero in $\mathbb{R}^d$. ∎

Proof of Theorem 2.4. The necessity of conditions (i) and (ii) in this theorem is immediate from Theorem 2.3. Moreover, it is sufficient, by Result 1.2, to prove sufficiency for suitability alone. So assume that $N_{\mathbb{R}}(\phi)$ is non-empty. We must show that A has either a π-periodic zero in $\mathbb{R}^d$ or a contaminating zero in $\mathbb{R}^d$.

To begin with, let Y be a minimal subset of Ξ for which $N_{\mathbb{R}}(\phi_Y)$ is not empty, i.e., fix some $Y \subset \Xi$ satisfying $N_{\mathbb{R}}(\phi_Y) \neq \{\ \}$, while $N_{\mathbb{R}}(\phi_X) = \{\ \}$ whenever $X \subsetneq Y$. Let U and V be as guaranteed by Corollary 3.1 and define

$$\hat{\phi}_V := \prod_{y \in Y} \hat{\phi}_u(\langle \cdot, v(y) \rangle).$$

This defines a compactly supported $\phi_V \in \mathscr{D}'(\mathbb{R}^{d_Y})$, which is refinable with mask

$$A_V := \prod_{y \in Y} A_y(\langle \cdot, v(y) \rangle).$$

We claim that $N_{\mathbb{R}}(\phi_V)$ is finite. If not, then Lemma 1.1 implies the existence of $\tilde{V} \subset V$ satisfying $\dim \operatorname{span} \tilde{V} < d_Y = \dim \operatorname{span} V$ (in particular, $\tilde{V} \neq V$) such that $N_{\mathbb{R}}(\phi_{\tilde{V}})$ is not empty, say $\eta \in N_{\mathbb{R}}(\phi_{\tilde{V}})$. (3.1.ii) implies that $\mathbb{R}^d U = \mathbb{R}^{d_Y}$, so there exists $\vartheta \in \mathbb{R}^d$ such that $\eta = \vartheta U$. Now define $X := U\tilde{V}$; then we have

$$\hat{\phi}_X(\vartheta + 2\alpha\pi) = \hat{\phi}_{\tilde{V}}(\eta + 2\alpha U\pi) = 0$$

for all $\alpha \in \mathbb{Z}^d$. The first equality follows from (3.1.i), and the second follows from (3.1.ii) together with the fact that η is in $N_{\mathbb{R}}(\phi_{\tilde{V}})$. But X is a proper subset of Y, which contradicts our choice of Y as minimal.

Next we show that $N_{\mathbb{R}}(\phi_V)$ is non-empty. Suppose that ϑ is in $N_{\mathbb{R}}(\phi_Y)$, i.e., that $\hat{\phi}_Y(\vartheta + 2\alpha\pi) = 0$ for all $\alpha \in \mathbb{Z}^d$. Let $\beta \in \mathbb{Z}^{d_Y}$ be arbitrary. Then (3.1)(ii) implies the existence of $\alpha \in \mathbb{Z}^d$ such that $\alpha U = \beta$. Therefore

$$\hat{\phi}_V(\vartheta U + 2\beta\pi) = \hat{\phi}_Y(\vartheta + 2\alpha\pi) = 0$$

by (3.1)(i). Since $\beta \in \mathbb{Z}^{d_Y}$ was arbitrary, we see that $N_{\mathbb{R}}(\phi_V)$ contains ϑU.

So, we may apply Theorem 2.1 or 2.2 to conclude that A_V has either a π-periodic zero in $\mathbb{R}^{d_Y}$ or a contaminating zero in $\mathbb{R}^{d_Y}$. If A_V contains a π-periodic zero, say $A_V(w + \beta\pi) = 0$ for all $\beta \in \mathbb{Z}^{d_Y}$, then, as above, there exists $z \in \mathbb{R}^d$ such that $zU = w$ by (3.1)(ii) and we have

$$A_Y(z + \alpha\pi) = A_V(w + \alpha U\pi) = 0$$

for all $\alpha \in \mathbb{Z}^d$, i.e., A_Y has a π-periodic zero. Since A_Y is a factor of A, we see that A has a π-periodic zero in $\mathbb{R}^d$ in this case.

Suppose now that A_V contains a contaminating zero in $\mathbb{R}^{d_Y}$. Let $\tilde{V} \subset V$ be a basis for $\mathbb{R}^{d_Y}$ and define $\tilde{Y} := U\tilde{V}$. Then $\tilde{Y} \subset Y$ is a basis for span Y. Since Ξ spans, we can choose $X \subset \Xi \backslash \tilde{Y}$, so that $X \cup \tilde{Y}$ forms a basis for $\mathbb{R}^d$. By Lemma 2.6, we may assume without loss of generality that this basis has determinant equal to ± 1. Since

$$(\tilde{Y} \quad X) = (U \quad X)\begin{pmatrix} \tilde{V} & 0 \\ 0 & I \end{pmatrix},$$

we must also have $|\det(U \quad X)| = 1$.

Now, let $m \in \{2, 3, 4, ...\}$ and $\eta \in \mathbb{Z}^{d_Y} \backslash (2^m - 1)\, \mathbb{Z}^{d_Y}$ be such that

$$A_V\left(2^k \frac{2\eta\pi}{2^m - 1} + \nu\pi\right) = 0 \qquad \text{for all} \quad \nu \in \mathbb{Z}^{d_Y} \backslash 2\mathbb{Z}^{d_Y},\ k \in \{0, 1, 2, ...\}.$$

Since $|\det(U \quad X)| = 1$, there exists $\mu \in \mathbb{Z}^d \backslash (2^m - 1)\, \mathbb{Z}^d$ such that $\mu U = \eta$, while $\langle \mu, x \rangle = 0$ for all $x \in X$. Moreover, for any $\nu \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d$ satisfying $\nu U \in 2\mathbb{Z}^{d_Y}$, there exists $x \in X$ for which $\langle \nu, x \rangle$ is odd. Therefore

$$A_{X \cup Y}\left(2^k \frac{2\mu\pi}{2^m - 1} + \nu\pi\right) = \prod_{x \in X} A_x(\langle \nu, x \rangle \pi)\, A_V\left(2^k \frac{2\eta\pi}{2^m - 1} + \nu U\pi\right) = 0. \quad \blacksquare$$

4. EXAMPLES

EXAMPLE 4.1. Our first example will show that the hypothesis dim span $\Xi = d$ is necessary to arrive at the characterizations in Theorem 2.4.

We analyze the distribution $\phi \in \mathscr{D}'(\mathbb{R}^2)$ defined by

$$\phi\colon \mathscr{D}(\mathbb{R}^2) \to \mathbb{C}\colon f \mapsto \langle \phi, f \rangle := \tfrac{1}{3} \int_0^3 f(t, t)\, dt$$

or equivalently,

$$\hat{\phi}(\omega_1, \omega_2) = \frac{e^{-3i(\omega_1 + \omega_2)} - 1}{-3i(\omega_1 + \omega_2)} \qquad \text{for} \quad (\omega_1, \omega_2) \in \mathbb{C}^2. \tag{4.1}$$

Equation (4.1) makes it easy to see that $\hat{\phi}((\pi/3, \pi/3) + 2\alpha\pi) = 0$ for every $\alpha \in \mathbb{Z}^d$; so the shifts of ϕ are neither linearly independent nor suitable.

This distribution is refinable and of type (1.2) with mask A_Ξ given by

$$\Xi := \{\xi\} := \left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} \qquad \text{and} \qquad A_\Xi = A_\xi(\langle \cdot, \xi \rangle) := \frac{e^{-3i\langle \cdot, \xi \rangle} + 1}{2}.$$

We will see that although the shifts of ϕ are not suitable, A_Ξ has no π-periodic zeros in $\mathbb{C}^2$ and no contaminating zeros in $\mathbb{R}^2$. Since $A_\xi(z\pi)$ is zero if and only if $3z$ is an odd integer, it will be sufficient to find $\nu_1, \nu_2 \in \{0, 1\}^2 \backslash 0$ such that $3\langle \nu_1, \xi \rangle$ is even while $3\langle \nu_2, \xi \rangle$ is odd. The choice $\nu_1 = (1, 1)$, $\nu_2 = (0, 1)$ will do.

We could build on this example to see that the assumption dim span $\Xi = d$ is still not sufficient without further assuming that $A_\xi(\pi) = 0$ for all $\xi \in \Xi$. Let $\Xi = \{\xi, \eta\}$ where ξ and A_ξ are as above and η is any vector in $\mathbb{Z}^2 \backslash \operatorname{span} \xi$ (so dim span $\Xi = d = 2$). Define $A_\eta(\omega) := \frac{1}{3} + \frac{2}{3} e^{-i\omega}$. Then A_η is a trigonometric polynomial with $A_\eta(0) = 1$ (hence it is the mask of some compactly supported refinable distribution ϕ_η). The shifts of ϕ_Ξ are not suitable (by the same reasoning as above), while it is clear that A_Ξ has neither π-periodic nor contaminating zeros in $\mathbb{R}^2$. (Since A_η has no real zeros, the real zeros of A_Ξ are as above.)

It is true that ϕ_η (and in fact ϕ_Ξ) in this construction is not of function type. But, as the remark following Corollary 2.5 indicates, the assumption $A_\xi(\pi)=0$ for all $\xi \in \Xi$ is not necessary when ϕ_ξ are all of function type.

EXAMPLE 4.2. Our next example will show that the sufficient conditions provided in Result 2.7 are not necessary in general.

For this example, we define the univariate mask

$$A_\theta := \frac{e^{-3i\cdot} + (1-2\cos\theta)\, e^{-2i\cdot} + (1-2\cos\theta)\, e^{-i\cdot} + 1}{4-4\cos\theta}$$

$$= \frac{(e^{-i\cdot}+1)(e^{-i\cdot}-e^{-i\theta})(e^{-i\cdot}-e^{i\theta})}{4-4\cos\theta}$$

for $\pi/3 < \theta < \pi$. Then A_θ is a trigonometric polynomial with real coefficients which satisfies $A_\theta(0)=1$. This is enough to imply the existence of a real-valued compactly supported refinable distribution with mask A_θ. There is a unique such distribution, ϕ_θ, if we insist further that $\hat{\phi}_\theta(0)=1$. In fact, for $\pi/3<\theta<\pi$, ϕ_θ is a continuous function with $\operatorname{supp}\phi_\theta=[0,3]$. The functions ϕ_θ are plotted for $\theta=15\pi/32$ and $\theta=17\pi/32$ in Fig. 1.

The zeros of the mask A_θ are $\{\pi, \theta, 2\pi-\theta\}+2\mathbb{Z}\pi$. From this we can see that the shifts of ϕ_θ are linearly independent for all $\theta \neq \pi/2$. We also see that $A_\theta(\pi)=0$.

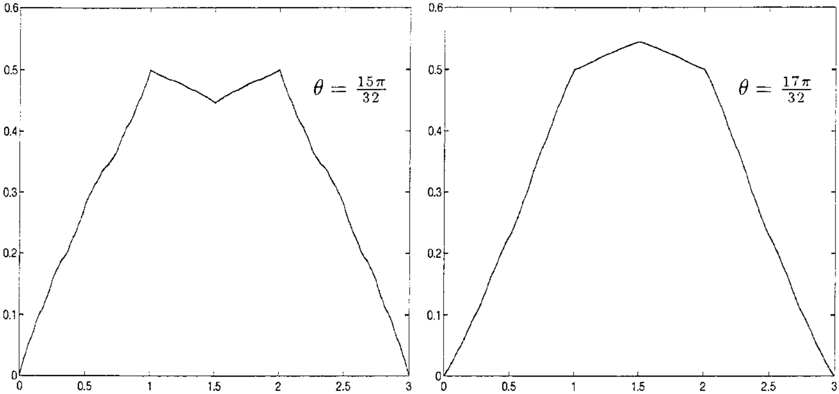

FIG. 1. Refinable functions with mast A_θ as described in Example 4.2.

In this example, we consider the bivariate function ϕ_Ξ of type (1.2) given by

$$\Xi = \{\xi, \eta, \zeta\} := \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\},$$

$$\phi_\xi = \phi_{15\pi/32}, \qquad \phi_\eta = \phi_{17\pi/32}, \qquad \text{and} \qquad \phi_\zeta = \chi_{[0,1)},$$

$$\hat{\phi}_\Xi = \hat{\phi}_\xi(\langle\cdot, \xi\rangle)\, \hat{\phi}_\eta(\langle\cdot, \eta\rangle)\, \hat{\phi}_\zeta(\langle\cdot, \eta\rangle)$$

which has mask

$$A_\Xi(\omega_1, \omega_2) = A_{15\pi/32}(\omega_1)\, A_{17\pi/32}(\omega_2) \left(\frac{e^{-i(\omega_1+\omega_2)}+1}{2}\right).$$

This defines a function $\phi := \phi_\Xi \in C^1(\mathbb{R}^2)$. Contour lines for this ϕ are shown in Fig. 2.

Each of the univariate functions has linearly independent shifts. Moreover, convolving any two also results in a function with linearly independent shifts. Also note that every basis $B \subset \Xi$ satisfies $|\det B| = 1$. However, the shifts of ϕ_Ξ are not even suitable, as you can see by observing that the zero set of A_Ξ consists of points $(x_1, x_2) \in \mathbb{R}^2$ satisfying

$$x_1 \in \left\{\pi, \frac{15\pi}{32}, \frac{49\pi}{32}\right\} + 2\mathbb{Z}\pi \qquad \text{or} \qquad x_2 \in \left\{\pi, \frac{17\pi}{32}, \frac{47\pi}{32}\right\} + 2\mathbb{Z}\pi$$

$$\text{or} \qquad x_1 + x_2 \in \pi + 2\mathbb{Z}\pi.$$

Thus A_Ξ has two π-periodic zeros: $A_\Xi((15\pi/32, 17\pi/32) + \alpha\pi) = 0$ and $A_\Xi((17\pi/32, 15\pi/32) + \alpha\pi) = 0$ for all $\alpha \in \mathbb{Z}^2$.

EXAMPLE 4.3. We provide an example of a box spline whose mask has a contaminating zero to generate some familiarity with contaminating zeros.

As far as this paper is concerned, it will be sufficient to define box splines in terms of their refinement mask. We can define a box spline M_Ξ associated with $\Xi \subset \mathbb{Z}^d \backslash 0$ by the refinement equation

$$\hat{M}_\Xi(2\,\cdot) = A_\Xi \hat{M}_\Xi, \qquad \text{where} \quad A_\Xi := \prod_{\xi \in \Xi} A_\xi(\langle\cdot, \xi\rangle) \qquad \text{and}$$

$$A_\xi := \left(\frac{1+e^{-i\cdot}}{2}\right)^{n_\xi}$$

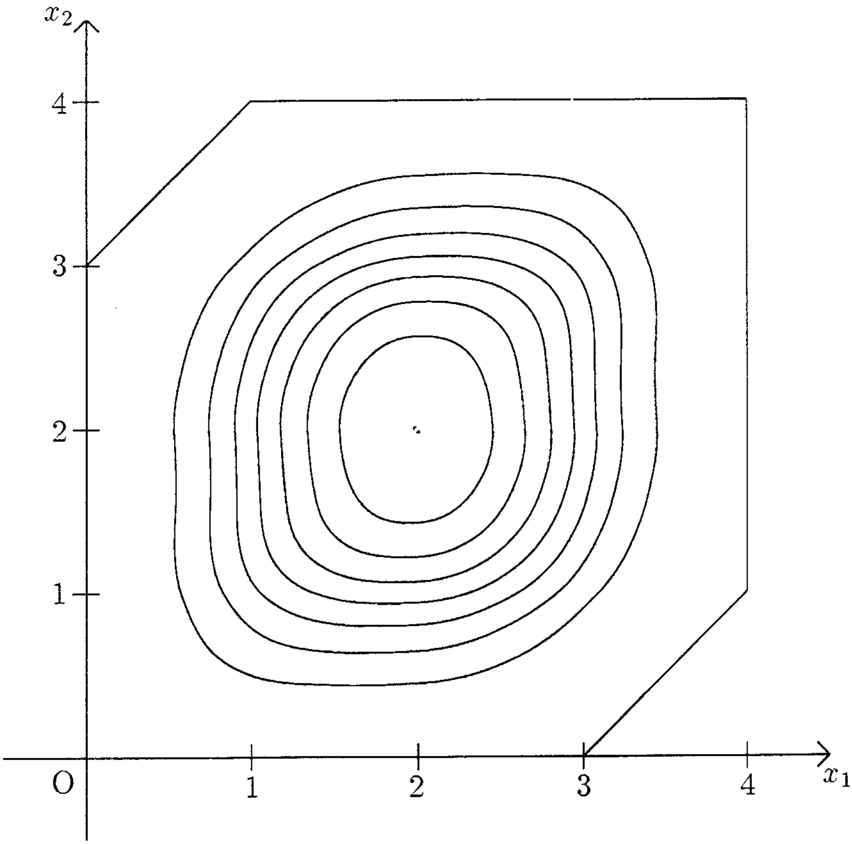

FIG. 2. Contour lines of the function ϕ from Example 4.2. The hexagon is the boundary of supp ϕ, the maxima are barely visible near the center, and the remaining contour lines are at equally spaced values of ϕ.

(along with $\hat{M}_{\Xi}(0) = 1$). Here, $n := (n_{\xi})_{\xi \in \Xi} \in \mathbb{N}^{\Xi}$ is the *multiplicity* of the *direction set* Ξ.

We have suppressed the dependence on the multiplicity n because the results of this paper are in terms of the zeros of the mask A_{Ξ}. It is clear that this set is independent of n. In fact, we see that

$$A_{\Xi}(z) = 0 \Leftrightarrow \langle x, \xi \rangle \in (\mathbb{Z} \backslash 2\mathbb{Z})\, \pi \qquad \text{for some} \quad \xi \in \Xi.$$

We should point out that, although our definition of box splines requires $\Xi \subset \mathbb{Z}^d$, the standard definition of box splines allows for arbitrary $\Xi \subset \mathbb{R}^d \backslash 0$. However, box splines with non-integer directions are, in general, not refinable. Hence we are not concerned with them here.

In this example, we let $d=2$ and we consider $\phi_\Xi := M_\Xi$, where

$$\Xi = \{\xi, \eta\} := \left\{ \begin{pmatrix} 6 \\ 5 \end{pmatrix}, \begin{pmatrix} -3 \\ 5 \end{pmatrix} \right\}.$$

Since $|\det[\xi \quad \eta]| = 45 \neq 1$, Result 2.7 implies that the shifts of ϕ_Ξ are not suitable. Indeed, ϕ_Ξ has a contaminating zero with $m=4$ and $\mu=(5,3)$. In Fig. 3 we have denoted the points $2^k(2\mu\pi/(2^m-1)) \in \mathbb{T}^d$ by bullets($\bullet$). The contaminating zero set is marked by asterisks($*$). We have also displayed particular curves $\langle \cdot, \xi \rangle \in (\mathbb{Z}\backslash 2\mathbb{Z})\,\pi$ and $\langle \cdot, \eta \rangle \in (\mathbb{Z}\backslash 2\mathbb{Z})\,\pi$ which cover this contaminating zero set.

EXAMPLE 4.4. Our final example involves box splines M_Ξ for which dim span Ξ is d. We will see that the necessary conditions for suitability provided in Result 2.7 are also sufficient for these functions. More specifically, the theorems of this paper provide a simple proof of the classic

Result 4.5 [6, 5]. For $\Xi \subset \mathbb{Z}^d \backslash 0$ with dim span $\Xi = d$, the shifts of any box spline M_Ξ associated with Ξ are linearly independent if and only if

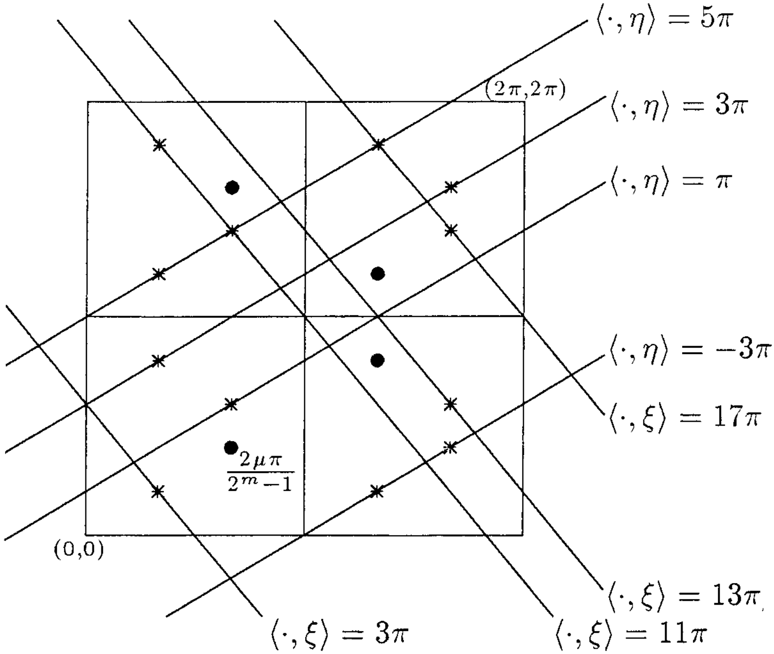

FIG. 3. Contaminating zero set with $2\mu\pi/(2^m-1) = (2\pi/3, 2\pi/5)$ from Example 4.3.

they are ℓ^p-stable for all $1 \leqslant p \leqslant \infty$ if and only if every basis in Ξ has determinant ± 1.

Linear combinations of shifts of box splines appear in the first paper on box splines, [1]. The fact that all bases B must satisfy $|\det B| = 1$ for linear independence was proved in [2]. The sufficiency of this condition was proved in [6] and, independently, in [5]. A more recent exposition of the theory of box splines is provided in [3].

Result 4.5 deals only with box splines for which $\dim \operatorname{span} \Xi = d$. Under this assumption, M_Ξ is a compactly supported function on $\mathbb{R}^d$. In fact, $M_\Xi \in L^p(\mathbb{R}^d)$ for $1 \leqslant p \leqslant \infty$. So, suitability implies stability for the shifts of M_Ξ. Also, since A_Ξ has only real zeros, stability is equivalent to linear independence. So we only need to prove Result 4.5 with regard to suitability.

Proof of Result 4.5. By Result 2.7, if the shifts of M_Ξ are suitable, then $|\det B| = 1$ for every basis $B \subset \Xi$. Now we must prove the converse. This is done using Theorem 2.4; i.e., we will show that if $A := A_\Xi$ has either a π-periodic zero in $\mathbb{R}^d$ or a contaminating zero in $\mathbb{R}^d$, then there exists a basis $B \subset \Xi$ with $|\det B| \neq 1$.

We begin by assuming that A has a π-periodic zero in $\mathbb{R}^d$; i.e., we assume the existence of $z \in \mathbb{R}^d$ so that for all $\alpha \in \mathbb{Z}^d$, there exists $\xi \in \Xi$ for which

$$\langle z + \alpha\pi, \xi \rangle \in (\mathbb{Z} \backslash 2\mathbb{Z})\, \pi.$$

We also assume that $|\det B| = 1$ for all bases $B \subset \Xi$ and arrive at a contradiction.

Let $Y \subset \Xi$ be a minimal subset of Ξ for which $A_Y(z + \alpha\pi) = 0$ for all $\alpha \in \mathbb{Z}^d$. Since $A_\xi(x) = 0 \Rightarrow x \in \mathbb{Z}\pi$ for any ξ, we must have $\langle z, y \rangle \in \mathbb{Z}\pi$ for every $y \in Y$. Moreover, the elements of Y must be linearly independent. For, suppose we have $a \in \mathbb{R}^Y \backslash 0$ for which

$$\sum_{y \in Y} y a(y) = 0.$$

Since $Y \subset \mathbb{Z}^d$, we may assume that $a \in \mathbb{Z}^Y \backslash 2\mathbb{Z}^Y$. Suppose $\tilde{y} \in Y$ is such that $a(\tilde{y})$ is odd. Then we have

$$a(\tilde{y}) \langle z + \alpha\pi, \tilde{y} \rangle = - \sum_{x \in Y \backslash \tilde{y}} a(x) \langle z + \alpha\pi, x \rangle.$$

We see that any α for which $\langle z + \alpha\pi, \tilde{y} \rangle \notin 2\mathbb{Z}\pi$ has the same property for some other element of Y. So z would actually be a π-periodic zero of $A_{Y \backslash \tilde{y}}$.

Now, Y is a subset of some basis B which, by assumption, satisfies $|\det B| = 1$. So $\mathbb{Z}^d B = \mathbb{Z}^d$. In particular, we can find $\alpha \in \mathbb{Z}^d$ so that

$$\langle \alpha, y \rangle = \begin{cases} 1 & \text{if } \langle z, y \rangle \in (\mathbb{Z} \backslash 2\mathbb{Z})\, \pi, \\ 0 & \text{otherwise.} \end{cases}$$

We see that, for this α, no $y \in Y$ satisfies $\langle z + \alpha\pi, y \rangle \in (\mathbb{Z} \backslash 2\mathbb{Z})\, \pi$, and this contradicts our assumptions.

Next, we assume that A_ξ has a contaminating zero in $\mathbb{R}^d$. I.e., we assume the existence of $m \in \{2, 3, ...\}$ and $\mu \in \mathbb{Z}^d \backslash (2^m - 1)\, \mathbb{Z}^d$ so that, for all $v \in \mathbb{Z}^d \backslash 2\mathbb{Z}^d$, $k \in \{0, 1, 2, ...\}$, there exists $\xi \in \Xi$ for which

$$\left\langle 2^k \frac{2\mu\pi}{2^m - 1} + v\pi, \xi \right\rangle \in (\mathbb{Z} \backslash 2\mathbb{Z})\, \pi.$$

We also assume that $|\det B| = 1$ for every basis $B \subset \Xi$ to arrive at another contradiction.

With $Y \subset \Xi$ minimally satisfying the condition that A_Y have a contaminating zero in $\mathbb{R}^d$, we must have $\langle \mu, y \rangle \in (2^m - 1)\, \mathbb{Z}$ for every $y \in Y$, and that the elements of Y be linearly independent.

If $\# Y = d$, then the elements of Y actually form a basis with determinant ± 1. But this is impossible, since $\mu \in \mathbb{Z}^d \backslash (2^m - 1)\, \mathbb{Z}^d$, while we must have $\langle \mu, y \rangle \in (2^m - 1)\, \mathbb{Z}$, for all $y \in Y$. On the other hand, if $\# Y < d$, then Y is a proper subset of some basis B which, by assumption, satisfies $|\det B| = 1$. We can therefore find $v \in \mathbb{Z}^d$ for which $\langle v, y \rangle$ is even for each $y \in Y$, while $v \notin 2\mathbb{Z}^d$. Since $\langle 2\mu/(2^m - 1), \alpha \rangle$ is an even integer for any $\alpha \in \mathbb{Z}^d$, we have $A_Y(2\mu\pi/(2^m - 1) + v\pi) \neq 0$. ∎

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